

Fast-forward scaling in a finite-dimensional Hilbert space

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Time evolution of quantum systems is accelerated by the fast-forward scaling. We reformulate the method to study systems in a finite-dimensional Hilbert space. For several simple systems, we explicitly construct the acceleration potential. We also use our formulation to accelerate the adiabatic dynamics. Applying the method to the transitionless quantum driving, we find that the fast-forward potential can be understood as a counterdiabatic term.

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I. INTRODUCTION

Recent developments in techniques of controlling quantum systems have brought about inventing new theoretical methods of acceleration. In the methods called the assisted adiabatic passage [1, 2] or the transitionless quantum driving [3], the acceleration of the adiabatic state is realized by applying the counterdiabatic Hamiltonian. The method is known to be equivalent to the Lewis-Riesenfeld invariant-based engineering [4, 5]. The optimal driving Hamiltonian is constructed under the condition that there exists an invariant quantity throughout the time evolution. Although the formulation is different between their methods, the essential mechanism is shown to be the same. We therefore call them generally “shortcuts to adiabaticity” [6]. Furthermore, these methods are derived from the quantum brachistochrone equation by imposing a proper constraint [7, 8], which shows that they admit a unified interpretation. The method has been implemented experimentally [9–12] and is important also for practical applications.

The fast-forward scaling method proposed by Masuda and Nakamura is known to be different from the other methods to design shortcuts to adiabaticity and plays a unique role in the methods of acceleration [13–16]. The method is formulated in the coordinate representation of the Schrödinger equation. It is applied to an evolving wave packet, either in noninteracting matter wave of a Bose-Einstein condensate described by the Gross-Pitaevski equation. The main idea is to determine the auxiliary accelerating Hamiltonian by a local potential term. To find the acceleration potential, the unitary transformation is performed for the original state.

From a more general perspective, we expect that there exists some relation between the fast-forward scaling method and the other techniques to design shortcuts to adiabaticity, mentioned above. In the fast-forward scaling, the form of the acceleration potential depends explicitly on the wavefunction to accelerate and takes a complicated form. The original formulation of the fast-forward method is limited to matter waves described as continuous variable systems. The extension of this method to finite-dimensional Hilbert space remains an interesting open problem. In Ref. [17], Masuda and Rice formulated

the fast-forward scaling in lattice systems aiming at the application to a Bose-Einstein condensate in an optical lattice. This formulation can be more generalized to treat other discrete systems such as spin models.

In this paper, we formulate the method in an arbitrary finite-dimensional Hilbert space. Although the main calculation treats two-level systems, it is a straightforward task to extend the formulation to higher-dimensional systems. Furthermore, we study how the method is related to the acceleration of the adiabatic state. Applying the fast-forward scaling to the formula of the transitionless driving, we show that the fast-forwarded state is also transitionless.

II. FAST-FORWARD SCALING

A. Formulation

We start from the time-dependent Schrödinger equation with the Hamiltonian $\hat{H}(t)$

$$i\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle. \quad (1)$$

For a given Hamiltonian, we assume that we have a solution of state $|\psi(t)\rangle$ in hand. We want to accelerate this time evolution by applying an external potential. In the fast-forward scaling, the time t is reparametrized as

$$\Lambda(t) = \int_0^t dt' \alpha(t'), \quad (2)$$

where $\alpha(t)$ is real and is greater than or equal to unity. In this new scale, the evolution of the state is fast-forwarded. Then, for a reason described below, we apply a unitary transformation

$$\hat{U}(t) = e^{-i\hat{f}(t)}, \quad (3)$$

where $\hat{f}(t)$ is a Hermitian operator. The Schrödinger equation is rewritten as

$$i\frac{d}{dt}|\psi_{\text{FF}}(t)\rangle = \hat{H}_{\text{FF}}(t)|\psi_{\text{FF}}(t)\rangle, \quad (4)$$

where

$$|\psi_{\text{FF}}(t)\rangle = \hat{U}(t)|\psi(\Lambda(t))\rangle, \quad (5)$$

$$\hat{H}_{\text{FF}}(t) = \hat{U}(t) \left(-i \frac{d}{dt} \hat{U}^\dagger(t) \right) + \alpha(t) \hat{U}(t) \hat{H}(\Lambda(t)) \hat{U}^\dagger(t). \quad (6)$$

The idea of the fast-forward scaling is to write the fast-forward Hamiltonian as the sum of the original Hamiltonian and the acceleration potential:

$$\hat{H}_{\text{FF}}(t) \sim \hat{H}(t) + \hat{V}(t). \quad (7)$$

The operators in the left- and right-hand sides are not equal to each other. The relation with the symbol \sim means that the operation to the state $|\psi_{\text{FF}}(t)\rangle$ gives the same effect as

$$\hat{H}_{\text{FF}}(t)|\psi_{\text{FF}}(t)\rangle = (\hat{H}(t) + \hat{V}(t))|\psi_{\text{FF}}(t)\rangle. \quad (8)$$

In the original study, the acceleration potential $\hat{V}(t)$ is represented by a local potential term. It can be possible by choosing the unitary transformation in a proper way.

We apply the method to discrete systems such as spin systems. As the simplest case, we use a two-level Hamiltonian

$$\begin{aligned} \hat{H}(t) &= \frac{1}{2} \mathbf{h}(t) \cdot \boldsymbol{\sigma} \\ &= \frac{1}{2} \begin{pmatrix} h_3(t) & h_1(t) - ih_2(t) \\ h_1(t) + ih_2(t) & -h_3(t) \end{pmatrix}, \end{aligned} \quad (9)$$

where $\mathbf{h}(t) = (h_1(t), h_2(t), h_3(t))$ is a three-dimensional magnetic-field vector and each component of $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ denotes a Pauli matrix.

First, we discuss how to choose the operator \hat{f} in Eq. (3). In the original analysis, f is a coordinate-dependent operator as $\hat{f} = f(\hat{x}, t)$. In the present two-level systems, anticipating that we measure the spin in z direction, we choose

$$\hat{f}(t) = \frac{\phi(t)}{2} \sigma_z, \quad (10)$$

where $\phi(t)$ is a real scalar function determined below. The probability that the state $|\psi(t)\rangle$ is observed in the up or down spin state $|\sigma = \pm 1\rangle$ is unchanged under the unitary transformation:

$$|\langle \sigma | \psi(t) \rangle|^2 = |\langle \sigma | e^{-i\phi(t)\sigma_z/2} | \psi(t) \rangle|^2. \quad (11)$$

We can also consider an operator $\hat{f}(t)$ which is proportional to the identity operator. It does not give any quantum effects and is not important. By using the gauge transformation, we can eliminate the identity-operator term in the acceleration potential as we mention below.

Second, under the choice of $\hat{f}(t)$ in Eq. (10), we separate the Hamiltonian into two parts:

$$\hat{H}_{\text{FF}}(t) = \frac{1}{2} (\mathbf{h}(t) \cdot \boldsymbol{\sigma} + \tilde{\mathbf{h}}(t) \cdot \boldsymbol{\sigma}), \quad (12)$$

$$\tilde{\mathbf{h}}(t) = \begin{pmatrix} \alpha \left(h_{1\Lambda} \cos \phi - h_{2\Lambda} \sin \phi \right) - h_1 \\ \alpha \left(h_{1\Lambda} \sin \phi + h_{2\Lambda} \cos \phi \right) - h_2 \\ \dot{\phi} + \alpha h_{3\Lambda} - h_3 \end{pmatrix}, \quad (13)$$

where $h_{1\Lambda} = h_1(\Lambda(t))$ and $\dot{\phi}$ is the time derivative of ϕ . The first term of $\hat{H}_{\text{FF}}(t)$ is the original Hamiltonian before the scaling. To write the second term by using σ_z only, we need to know the explicit form of the wavefunction. We write the original state as

$$|\psi(t)\rangle = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = a(t)|+\rangle + b(t)|-\rangle. \quad (14)$$

The “coordinate” representation of the state $\psi_{\text{FF}}(\sigma, t) = \langle \sigma | \psi_{\text{FF}}(t) \rangle$ is

$$\begin{aligned} \psi_{\text{FF}}(\sigma, t) &= \begin{Bmatrix} e^{-i\phi/2} a_\Lambda \\ e^{i\phi/2} b_\Lambda \end{Bmatrix} \\ &= e^{-i\phi/2} a_\Lambda \frac{1+\sigma}{2} + e^{i\phi/2} b_\Lambda \frac{1-\sigma}{2}, \end{aligned} \quad (15)$$

where $a_\Lambda = a(\Lambda(t))$ and $b_\Lambda = b(\Lambda(t))$. The symbol σ takes ± 1 , which plays the role of “ x ”. In the same way, we can write

$$\begin{aligned} \langle \sigma | \sigma_z | \psi_{\text{FF}} \rangle &= \begin{Bmatrix} e^{-i\phi/2} a_\Lambda \\ -e^{i\phi/2} b_\Lambda \end{Bmatrix} \\ &= \sigma \psi_{\text{FF}}, \end{aligned} \quad (16)$$

$$\begin{aligned} \langle \sigma | \sigma_x | \psi_{\text{FF}} \rangle &= \begin{Bmatrix} e^{i\phi/2} b_\Lambda \\ e^{-i\phi/2} a_\Lambda \end{Bmatrix} \\ &= \left(\frac{b_\Lambda}{a_\Lambda} e^{i\phi} \frac{1+\sigma}{2} + \frac{a_\Lambda}{b_\Lambda} e^{-i\phi} \frac{1-\sigma}{2} \right) \psi_{\text{FF}}, \end{aligned} \quad (17)$$

$$\begin{aligned} \langle \sigma | \sigma_y | \psi_{\text{FF}} \rangle &= \begin{Bmatrix} -ie^{i\phi/2} b_\Lambda \\ ie^{-i\phi/2} a_\Lambda \end{Bmatrix} \\ &= -i \left(\frac{b_\Lambda}{a_\Lambda} e^{i\phi} \frac{1+\sigma}{2} - \frac{a_\Lambda}{b_\Lambda} e^{-i\phi} \frac{1-\sigma}{2} \right) \psi_{\text{FF}}. \end{aligned} \quad (18)$$

Using these relations, we obtain the acceleration potential $V(t) = \text{diag}(V(+1, t), V(-1, t))$ where

$$\begin{aligned} V(\sigma, t) &= \frac{1}{2} (\tilde{h}_1 - i\tilde{h}_2) \frac{b_\Lambda}{a_\Lambda} e^{i\phi} \frac{1+\sigma}{2} \\ &\quad + \frac{1}{2} (\tilde{h}_1 + i\tilde{h}_2) \frac{a_\Lambda}{b_\Lambda} e^{-i\phi} \frac{1-\sigma}{2} + \frac{1}{2} \tilde{h}_3 \sigma. \end{aligned} \quad (19)$$

The value of $\phi(t)$ is determined so that the potential is real. The condition is given by

$$(\tilde{h}_1 - i\tilde{h}_2) \frac{b_\Lambda}{a_\Lambda} e^{i\phi} = (\tilde{h}_1 + i\tilde{h}_2) \frac{b_\Lambda^*}{a_\Lambda^*} e^{-i\phi}. \quad (20)$$

Thus the acceleration potential has the form $V(\sigma, t) = (v_0(t) + v(t)\sigma)/2$ with real $v_0(t)$ and $v(t)$. Since the term $v_0(t)$ only affects the overall phase of the state, it does not play any role for the acceleration. We can eliminate this term by using a unitary transformation (3) with the form $\hat{f}(t) = (\phi_0(t) + \phi(t)\sigma_z)/2$.

B. Example: two-level system

To see how the method works, we treat an example of a two-level system. The magnetic field is chosen to be

$$\mathbf{h}(t) = \begin{pmatrix} h_0 \cos \omega t \\ -\omega \\ h_0 \sin \omega t \end{pmatrix}. \quad (21)$$

For simplicity, we set $h_0 = 1$ and make all variables dimensionless in the following calculations. The Hamiltonian reads

$$\hat{H}(t) = \frac{1}{2} \begin{pmatrix} \sin \omega t & \cos \omega t + i\omega \\ \cos \omega t - i\omega & -\sin \omega t \end{pmatrix}. \quad (22)$$

One of the exact solutions of the Schrödinger equation is given by

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-it/2} \begin{pmatrix} \cos \frac{\omega t}{2} + \sin \frac{\omega t}{2} \\ \cos \frac{\omega t}{2} - \sin \frac{\omega t}{2} \end{pmatrix}. \quad (23)$$

We set the initial state at $t = 0$ is given by the eigenstate of σ_x with the eigenvalue $+1$. With the time evolution, the spin rotates around the y axis and points to the positive- z direction at $t = t_f = \pi/2\omega$. We are interested in fast-forwarding this motion.

The condition (20) is explicitly written as

$$\omega \alpha(t) - \cos \omega t \sin \phi(t) = \omega \cos \phi(t). \quad (24)$$

Using the obtained function $\phi(t)$, we can write the potential

$$V(\sigma, t) = \frac{1}{2} \left(\alpha + \frac{\omega \sin \phi - \cos \phi \cos \omega t}{\cos \omega \Lambda} \right) + \frac{1}{2} \left[\dot{\phi} - \frac{(\omega \sin \phi - \cos \phi \cos \omega t) \sin \omega \Lambda}{\cos \omega \Lambda} - \sin \omega t \right] \sigma. \quad (25)$$

Following the original analysis [13], we choose the magnification factor $\alpha(t)$ as

$$\alpha(t) = \begin{cases} \bar{\alpha} + (1 - \bar{\alpha}) \cos \left(\frac{2\pi t}{t_0} \right) & 0 \leq t \leq t_0 \\ 1 & t > t_0 \end{cases}, \quad (26)$$

where $\bar{\alpha} > 1$. $\Lambda(t)$ is given by

$$\Lambda(t) = \begin{cases} \bar{\alpha} t + (1 - \bar{\alpha}) \frac{\sin \left(\frac{2\pi t}{t_0} \right)}{\frac{2\pi}{t_0}} & 0 \leq t \leq t_0 \\ \bar{\alpha} t_0 + t - t_0 & t > t_0 \end{cases}. \quad (27)$$

These functions are plotted in Fig. 1 for $\bar{\alpha} = 2$ and $t_0 = 10$. The final time $t_f = \pi/2\omega = 20$ before the fast-forwarding is shortened as 10 by the scaling. Correspondingly, the behavior of the acceleration potential $V(\sigma, t)$ is plotted as in Fig. 2. We see that the potential $V(\sigma = -1, t)$ diverges at the final time $t = 10$. This behavior is understood in the general expression of the potential in Eq. (19) where one of the components of the state a_Λ in the denominator goes to zero.

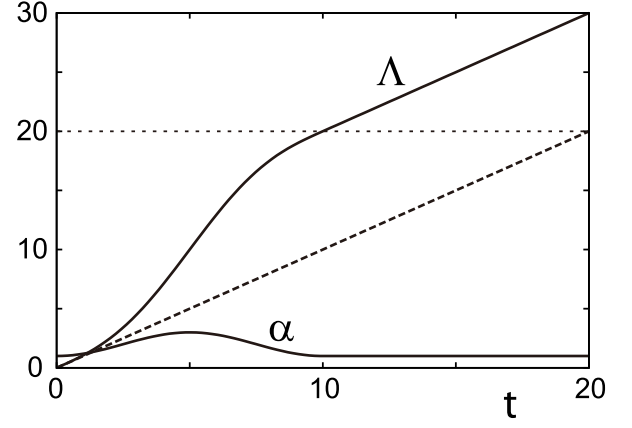


FIG. 1. Protocols (26) and (27) at $\bar{\alpha} = 2.0$ and $t_0 = 10$. The dashed line represents the time before the scaling.

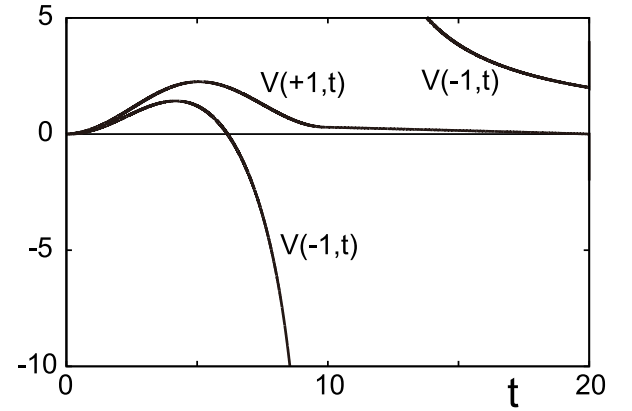


FIG. 2. Acceleration potential $V(\sigma = \pm 1, t)$ in Eq. (25). $V(-1, t)$ at $t = 10$ goes to $-\infty$ from the left and ∞ from the right.

The problem of the wave-function node was recognized in the original studies [14, 15]. It was discussed that the robustness against the potential variation holds if the phase $\phi(t)$ is not divergent. The divergence of the control Hamiltonian is also seen in the transitionless quantum driving. In that case, the divergence is due to the level crossing and leads to a serious problem. As for the present case, the divergence is considered to be a fictitious singularity which is not directly connected to any physical disaster. It appears when we try to represent the potential by the σ_z operator. The original Hamiltonian $\hat{H}_{\text{FF}}(t)$ does not include any singularity.

Figure 3 shows the numerical result of the probability of the up-spin state $|\langle \sigma = +1 | \psi_{\text{FF}}(t) \rangle|^2$. We can reach the final state at $t = 20$ before the scaling in a shorter time $t = 10$. The numerical result agrees with the analytical one very well.

In the above example, Eq. (24) is solved safely to find a real $\phi(t)$. Generally, the condition (20) does not always have a solution. This is understood from the following

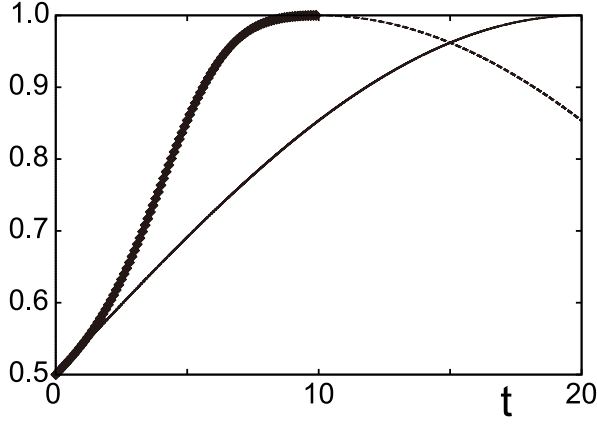


FIG. 3. Probability of the up-spin state $|\langle\sigma = 1|\psi(t)\rangle|^2$. The solid line represents the result without the fast-forward scaling. The bold and dashed lines represent the numerical and analytical results with the scaling, respectively. $\bar{\alpha} = 2.0$; $t_0 = 10$.

example. We consider the magnetic field

$$\mathbf{h}(t) = \begin{pmatrix} h(t) \cos \omega t \\ h(t) \sin \omega t \\ \omega \end{pmatrix}. \quad (28)$$

When we take the down-spin state $|- \rangle$ as the initial one $|\psi(0)\rangle$, the time dependence of the state is given by

$$|\psi(t)\rangle = \begin{pmatrix} -ie^{-i\omega t/2} \sin\left(\frac{1}{2} \int_0^t dt' h(t')\right) \\ e^{i\omega t/2} \cos\left(\frac{1}{2} \int_0^t dt' h(t')\right) \end{pmatrix}. \quad (29)$$

The condition (20) is written as

$$\alpha(t)h(\Lambda(t)) = h(t) \cos(\phi(t) + \omega\Lambda(t) - \omega t). \quad (30)$$

Since $\alpha(t) \geq 1$, the solution can be found only when $h(t)$ is a decreasing function.

C. Generalization

It is straightforward to extend the method to systems in an N -dimensional Hilbert space. In this case, we can choose the unitary transformation

$$\hat{U}(t) = \exp\left(-i \sum_{a=1}^{N-1} \phi_a(t) \hat{X}_a\right). \quad (31)$$

The operators $\{\hat{X}_a\}_{a=1,2,\dots,N-1}$ are traceless and commute with each other:

$$[\hat{X}_a, \hat{X}_b] = 0. \quad (32)$$

In the N -dimensional Hilbert space, there exist $N - 1$ independent diagonal traceless matrices. For $N = 2$, we have only one operator $\hat{X} = \sigma_z/2$ as we have already discussed.

In the N -dimensional case, the acceleration potential takes the form

$$\hat{V}(t) = \frac{1}{N} v_0(t) + \sum_{a=1}^{N-1} v_a(t) \hat{X}_a. \quad (33)$$

The first term is proportional to the identity operator. We choose the coefficients $\{v_0(t), v_1(t), \dots, v_{N-1}(t)\}$ so that the condition $\hat{H}_{\text{FF}}(t) \sim \hat{H}(t) + \hat{V}(t)$ holds.

D. Example: two-spin system

To see how the generalization described above works well, we consider the second example in a two-spin system where the Hilbert space has four dimensions. We consider the Hamiltonian

$$\begin{aligned} \hat{H}(t) = & \sigma_z^{(1)} \sigma_z^{(2)} \sin \omega t - \frac{1}{2} (\sigma_x^{(1)} + \sigma_x^{(2)}) \cos \omega t \\ & + \frac{i\omega}{4} (\sigma_y^{(1)} \sigma_z^{(2)} + \sigma_z^{(1)} \sigma_y^{(2)}), \end{aligned} \quad (34)$$

where $\sigma^{(1,2)}$ denote the Pauli matrices for spins 1 and 2, respectively. This is an example used in Ref. [18] to study the transitionless driving. The last term of the Hamiltonian corresponds to the counterdiabatic term. One of the solutions of the Schrödinger equation is written as

$$|\psi(t)\rangle = \frac{e^{it}}{2\sqrt{1 + \sin \omega t}} \begin{pmatrix} \cos \omega t \\ 1 + \sin \omega t \\ 1 + \sin \omega t \\ \cos \omega t \end{pmatrix}, \quad (35)$$

where we take the basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ using the notations $\sigma_z |\uparrow\rangle = |\uparrow\rangle$ and $\sigma_z |\downarrow\rangle = -|\downarrow\rangle$. This state makes the transition from $|\psi(0)\rangle = |\rightarrow\rightarrow\rangle$, both spins pointing in the x direction, to an entangled state $|\psi(t_f = \pi/2\omega)\rangle = (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$.

We use the unitary transformation (3) with

$$\hat{f}(t) = \phi_1(t) \sigma_z^{(1)} + \phi_2(t) \sigma_z^{(2)} + \phi_3(t) \sigma_z^{(1)} \sigma_z^{(2)}. \quad (36)$$

Correspondingly, the acceleration potential takes the form

$$\hat{V}(t) = v_0(t) + v_1(t) \sigma_z^{(1)} + v_2(t) \sigma_z^{(2)} + v_3(t) \sigma_z^{(1)} \sigma_z^{(2)}. \quad (37)$$

We use the same protocol as the example in Sec. II B. Following the same manipulation, we obtain the conditions for $\phi_{1,2,3}(t)$ as

$$\phi_1(t) = \phi_2(t) = 0, \quad (38)$$

$$2 \cos \omega t \sin 2\phi_3(t) = \alpha(t) \omega. \quad (39)$$

Using $\phi_3(t)$, we can write the potential as

$$v_0(t) = -1 + \frac{\cos \omega t \cos 2\phi_3(t)}{\cos \omega \Lambda(t)}, \quad (40)$$

$$v_1(t) = v_2(t) = 0, \quad (41)$$

$$\begin{aligned} v_3(t) = & \dot{\phi}_3(t) + \alpha(t) \sin \omega \Lambda(t) - \sin \omega t \\ & - \sin \omega \Lambda(t) + \tan \omega \Lambda(t) \cos \omega t \cos 2\phi_3(t). \end{aligned} \quad (42)$$

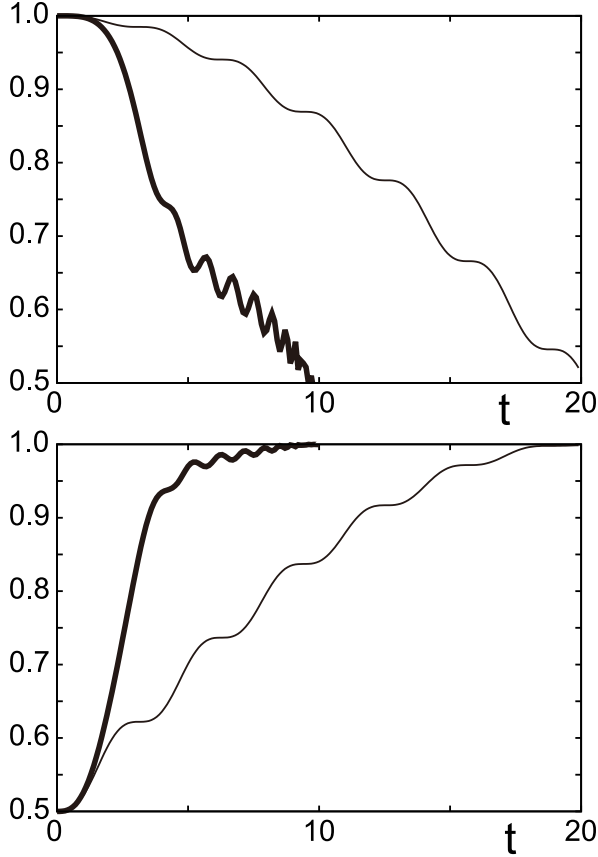


FIG. 4. Overlap of the state with the initial state $|\langle\psi(0)|\psi(t)\rangle|^2$ (upper panel) and with the final state $|\langle\psi(t_f)|\psi(t)\rangle|^2$ (lower). The solid line represents the result without the fast-forward scaling and the bold with the scaling. $\bar{\alpha} = 2.0$; $t_0 = 10$.

In the present example, the system can be accelerated by controlling the exchange interaction in a proper way as we show the numerical calculation in Fig. 4.

III. ACCELERATION OF ADIABATIC STATES

A. Transitionless quantum driving

We have demonstrated that the fast-forward scaling can be applied to systems in a finite-dimensional space. In this section, we consider the acceleration of adiabatic states. In the transitionless quantum driving, the Hamiltonian consists of the adiabatic and counterdiabatic terms: $\hat{H}(t) = \hat{H}_{\text{ad}}(t) + \hat{H}_{\text{cd}}(t)$. Each term is given respectively by

$$\hat{H}_{\text{ad}}(t) = \sum_n E_n(t) |n(t)\rangle \langle n(t)|, \quad (43)$$

$$\hat{H}_{\text{cd}}(t) = i \sum_{m \neq n} |m(t)\rangle \langle m(t)| \dot{n}(t) \langle n(t)|, \quad (44)$$

where $|n(t)\rangle$ is the eigenstate of $\hat{H}_{\text{ad}}(t)$ with the eigenvalue $E_n(t)$. The eigenstates are orthonormalized and satisfy the completeness relation. It can be shown that the solution of the Schrödinger equation is given by the adiabatic state of $\hat{H}_{\text{ad}}(t)$ [1–3].

We apply the fast-forward scaling to the transitionless driving. To achieve this, we first use the unitary transformation (31). Then, the fast-forward Hamiltonian is written as

$$\hat{H}_{\text{FF}}(t) = \sum_a \dot{\phi}_a(t) \hat{X}_a + \hat{U}(t) \hat{H}(\Lambda(t)) \hat{U}^\dagger(t). \quad (45)$$

Using the new basis

$$|\tilde{n}(t)\rangle = \hat{U}(t) |n(\Lambda(t))\rangle, \quad (46)$$

we can write $\hat{H}_{\text{FF}}(t) = \hat{H}_{\text{FF}}^{\text{ad}}(t) + \hat{H}_{\text{FF}}^{\text{cd}}(t)$ with

$$\begin{aligned} \hat{H}_{\text{FF}}^{\text{ad}}(t) &= \sum_n \left(\alpha(t) E_n(\Lambda(t)) + \sum_a \dot{\phi}_a(t) \langle \tilde{n}(t) | \hat{X}_a | \tilde{n}(t) \rangle \right) \\ &\quad \times |\tilde{n}(t)\rangle \langle \tilde{n}(t)|, \end{aligned} \quad (47)$$

$$\hat{H}_{\text{FF}}^{\text{cd}}(t) = i \sum_{m \neq n} |\tilde{m}(t)\rangle \langle \tilde{m}(t) | \dot{\tilde{n}}(t) \rangle \langle \tilde{n}(t)|. \quad (48)$$

This expression denotes that the fast-forward state of the transitionless driving is also transitionless. We note that the first term of Eq. (45) does not affect the counterdiabatic part.

Next, we determine the fast-forward potential $\hat{V}(t)$ such that the relation $\hat{H}_{\text{FF}}(t) \sim \hat{H}(t) + \hat{V}(t)$ holds. We consider the case where the state is in a specific eigenstate $|\tilde{n}(t)\rangle$. We impose the condition

$$\hat{H}_{\text{FF}}(t) |\tilde{n}(t)\rangle = (\hat{H}(t) + \hat{V}(t)) |\tilde{n}(t)\rangle \quad (49)$$

to find the potential in a diagonal form (33). The coefficients $\{v_0, v_1, \dots, v_{N-1}\}$ are determined from

$$\langle \tilde{n} | \hat{H}(t) | \tilde{n} \rangle = \alpha E_{n\Lambda} - \frac{1}{N} v_0 + \sum_a (\dot{\phi}_a - v_a) \langle \tilde{n} | \hat{X}_a | \tilde{n} \rangle, \quad (50)$$

$$\langle \tilde{m} | \hat{H}(t) | \tilde{n} \rangle = i \langle \tilde{m} | \dot{\tilde{n}} \rangle - \sum_a v_a \langle \tilde{m} | \hat{X}_a | \tilde{n} \rangle, \quad (51)$$

where $E_{n\Lambda} = E_n(\Lambda(t))$ and m takes values different from n . There are N -independent equations. The equations imply that the solution depends on the state n to use.

We examine the two-level case where $\hat{X} = \sigma_z/2$. The transitionless driving is achieved by the Hamiltonian

$$\hat{H}(t) = \frac{1}{2} \left(\mathbf{h}(t) + \frac{\mathbf{h}(t) \times \dot{\mathbf{h}}(t)}{h^2(t)} \right) \cdot \boldsymbol{\sigma}, \quad (52)$$

where the second term denotes the counterdiabatic field [1–3]. Using the polar coordinate representation of the magnetic field

$$\mathbf{h}(t) = h(t) \begin{pmatrix} \sin \theta(t) \cos \varphi(t) \\ \sin \theta(t) \sin \varphi(t) \\ \cos \theta(t) \end{pmatrix}, \quad (53)$$

we can write the eigenstates of $\hat{H}_{\text{ad}}(t) = \mathbf{h}(t) \cdot \boldsymbol{\sigma}/2$ as

$$|n(t)\rangle = \left\{ \begin{pmatrix} \cos \frac{\theta(t)}{2} \\ e^{i\varphi(t)} \sin \frac{\theta(t)}{2} \end{pmatrix}, \begin{pmatrix} -e^{-i\varphi(t)} \sin \frac{\theta(t)}{2} \\ \cos \frac{\theta(t)}{2} \end{pmatrix} \right\}. \quad (54)$$

We consider an acceleration of the former state under the choice of parameters

$$h(t) = 1, \quad (55)$$

$$\theta(t) = \frac{\pi}{2} - \omega t, \quad (56)$$

$$\varphi(t) = 0. \quad (57)$$

This is the example treated in Sec. II B. Equations (50) and (51) are written as

$$\begin{aligned} & \alpha - v_0 + (\dot{\phi} - v) \sin \omega \Lambda \\ &= \sin \omega t \sin \omega \Lambda + \cos \omega t \cos \omega \Lambda \cos \phi - \omega \cos \omega \Lambda \sin \phi, \end{aligned} \quad (58)$$

$$\begin{aligned} & i\alpha\omega + (\dot{\phi} - v) \cos \omega \Lambda \\ &= \sin \omega t \cos \omega \Lambda - \cos \omega t \sin \omega \Lambda \cos \phi \\ &+ i \cos \omega t \sin \phi + i\omega \cos \phi + \omega \sin \omega \Lambda \sin \phi, \end{aligned} \quad (59)$$

which gives the condition (24) and the potential (25).

Thus we find that the fast-forward scaling is useful when accelerating the adiabatic state. We note that the original fast-forward Hamiltonian $\hat{H}_{\text{FF}}(t) = \hat{H}_{\text{FF}}^{\text{ad}}(t) + \hat{H}_{\text{FF}}^{\text{cd}}(t)$ in Eq. (45) is enough to accelerate the state evolution. The advantage of the fast-forward scaling is that we can take the form of the acceleration potential in a diagonal form. This arbitrariness of the counterdiabatic Hamiltonian comes from the fact that the acceleration potential depends on the state to accelerate. The counterdiabatic Hamiltonian (44) is applied to any states $\{|n(t)\rangle\}$. If we consider a specific state $|n(t)\rangle$ only, it is possible to deform the counterdiabatic Hamiltonian as

$$\begin{aligned} \hat{H}_{\text{cd}}^{(n)}(t) &= i(1 - |n(t)\rangle\langle n(t)|)|\dot{n}(t)\rangle\langle n(t)| + (\text{H.c.}) \\ &+ (n\text{-independent terms}), \end{aligned} \quad (60)$$

where n -independent terms can be taken arbitrarily. This property was used in Refs. [19, 20] to apply the method in many-body systems where the counterdiabatic Hamiltonian (44) takes a complicated form. In Ref. [18], the same idea was used to obtain the partial suppression of the nonadiabatic transitions. In the present case, using the arbitrariness of the counterdiabatic Hamiltonian, we choose the acceleration potential in a diagonal form.

B. Lewis-Riesenfeld invariant

It is also possible to formulate the fast-forward scaling by using the invariant-based engineering. We consider the operator $\hat{F}(t)$ satisfying

$$i \frac{\partial \hat{F}(t)}{\partial t} = [\hat{H}(t), \hat{F}(t)]. \quad (61)$$

This operator is called the Lewis-Riesenfeld invariant [4] and the eigenvalues λ_n are independent of time:

$$\hat{F}(t) = \sum_n \lambda_n |n(t)\rangle\langle n(t)|, \quad (62)$$

where $|n(t)\rangle$ represents the corresponding eigenstate. In the invariant-based engineering, we construct the Hamiltonian for a given invariant [5]. Using the basis of the eigenstates of $\hat{F}(t)$, we can write the Hamiltonian as $\hat{H}(t) = \hat{H}_{\text{ad}}(t) + \hat{H}_{\text{cd}}(t)$, where each term is given by Eqs. (43) and (44), respectively. Thus the time evolution becomes transitionless. For example, in the case of the Hamiltonian in Sec. II B, the invariant is given by

$$\begin{aligned} \hat{F}(t) &= \frac{\lambda_+}{2} \begin{pmatrix} 1 + \sin \omega t & \cos \omega t \\ \cos \omega t & 1 - \sin \omega t \end{pmatrix} \\ &+ \frac{\lambda_-}{2} \begin{pmatrix} 1 - \sin \omega t & -\cos \omega t \\ -\cos \omega t & 1 + \sin \omega t \end{pmatrix}. \end{aligned} \quad (63)$$

This is obtained by substituting the form of the eigenstates in Eq. (54) to Eq. (62).

We apply the fast-forward scaling to the equation for the invariant (61). First, using the scaling $\Lambda(t)$ and the unitary transformation (31), we obtain

$$i \frac{\partial \hat{F}_{\text{FF}}(t)}{\partial t} = [\hat{H}_{\text{FF}}(t), \hat{F}_{\text{FF}}(t)], \quad (64)$$

where $\hat{H}_{\text{FF}}(t)$ is given in Eq. (45) and

$$\hat{F}_{\text{FF}}(t) = \hat{U}(t) \hat{F}(\Lambda(t)) \hat{U}^\dagger(t) = \sum_n \lambda_n |\tilde{n}(t)\rangle\langle \tilde{n}(t)|. \quad (65)$$

Second, we determine the acceleration potential $\hat{V}(t)$ in the form of Eq. (33) satisfying

$$i \frac{\partial \hat{F}_{\text{FF}}(t)}{\partial t} = [\hat{H}(t) + \hat{V}(t), \hat{F}_{\text{FF}}(t)]. \quad (66)$$

As we explained in the previous subsection, the solution depends on the state to accelerate. In the present case, the choice of the state is reflected in the eigenvalues $\{\lambda_n\}$. Here we consider the simplest case where one of the eigenvalues is one and the others are zero:

$$\hat{F}(t) = |n(t)\rangle\langle n(t)|. \quad (67)$$

From Eq. (66), we have

$$i \langle \tilde{m} | \dot{\tilde{n}} \rangle = \langle \tilde{m} | \hat{H}(t) | \tilde{n} \rangle + \sum_a v_a \langle \tilde{m} | \hat{X}_a | \tilde{n} \rangle, \quad (68)$$

where m takes values different from n . This equation coincides with Eq. (51). The diagonal part of the Hamiltonian cannot be determined from Eq. (66) since it does not contribute to the equation. We impose

$$\langle \tilde{n} | (\hat{H}(t) + \hat{V}(t)) | \tilde{n} \rangle = \langle \tilde{n} | \hat{H}_{\text{FF}}(t) | \tilde{n} \rangle. \quad (69)$$

This is equivalent to Eq. (50). Thus, in the present formulation based on the invariant, we find the same result as the formulation using the transitionless driving. The derivation denotes that the form of the acceleration potential depends on the choice of the eigenvalues of the invariant. It corresponds to setting the initial condition for the time evolution.

IV. SUMMARY

In summary, we have shown that the method of the fast-forward scaling is applicable to systems in a finite-dimensional Hilbert space. The unitary transformation (31) is utilized to have the acceleration potential in a diagonal form.

Although the use of the fast-forward scaling is not restricted to the acceleration of the adiabatic state, we find that the method is most useful when it is applied to the transitionless driving. In that case, the fast-forward state follows a different adiabatic passage and is understood as a different transitionless driving. The advantage of using the transitionless driving is that the general condition to determine the acceleration potential can be explicitly written as (50) and (51). Using both methods together, we can consider the efficient acceleration of the state.

The form of the acceleration potential is state dependent. Different states require different auxiliary fast-forward driving potentials. This limitation is absent in the transitionless driving which applies to an arbitrary state. Furthermore, our analysis implies that the accel-

eration is not always possible. The condition of determining the unitary transformation sometimes fails to find the acceleration potential.

As a related problem, it will be interesting to know the relation of the fast-forward scaling to the method proposed in Ref. [21]. Using the framework of transitionless quantum driving, the counterdiabatic driving was derived for a large family of many-body and nonlinear systems under scale-invariant dynamics. It may be interesting to study the fast-forward scaling in such systems, and to explore the prospects of extending the fast-forward technique beyond single-particle and mean-field descriptions.

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